TUT 4: SOLVE A SECOND ORDER LINEAR EQUATION

YUAN CHEN

For your convenience, I use "t" as independent variable from now on.

Q1. Solve the following homogenous second order equations.

(a) $y'' + 2y' - 3y = 0$;
(b) $y'' + 4y' + 4y = 0$;
(c) $y'' + y' + 4y = 0$.

Solution. (a) By the characteristic equation

$$\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0, \qquad \text{two real roots}$$

and hence $\{e^t,\, e^{-3t}\}$ is the fundamental set of solutions. Therefore the general solution is

$$y = c_1 e^t + c_2 e^{-3t}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

(b) By the characteristic equation

$$\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0, \qquad \text{one real root with multiplicity 2}$$

and hence $\{e^{-2t},\, te^{-2t}\}$ is the fundamental set of solutions. Therefore the general solution is

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

(c) By the characteristic equation

$$\lambda^2 + \lambda + 4 = 0, \quad \text{and then} \quad \lambda = \frac{-1 \pm \sqrt{1 - 4 \times 4}}{2} = \frac{-1 \pm \sqrt{15}i}{2}.$$

Hence

$$e^{-t/2} \cos \frac{\sqrt{15}}{2} t \quad \text{and} \quad e^{-t/2} \sin \frac{\sqrt{15}}{2} t$$

forms a fundamental set of solutions. Therefore the general solution is

$$y = c_1 e^{-t/2} \cos \frac{\sqrt{15}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{15}}{2} t$$

for some constants $c_1, c_2 \in \mathbb{R}$. □

Q2. Solve the following non-homogeneous equations:

(a) $y'' + 2y' - 3y = e^{-2t} + e^t$;
(b) $y'' + 4y' + 4y = e^{-2t} + t^2$;
(c) $y'' + 4y = \cos 3t + \sin 2t$;
(d) $y'' + 2y' - 3y = e^{2t} \sin 2t + te^t + t \sin t$.

Solution. (a) Step 1. (*Solve the homogenous eq*) By **Q1**(a), we know $S := \{e^t, e^{-3t}\}$ is the fundamental set of solutions to the homogeneous equation.

Step 2. (*Find a particular solution*) Consider

$$\begin{cases} Y_1'' + 2Y_1' - 3Y_1 = e^{-2t}; \\ Y_2'' + 2Y_2' - 3Y_2 = e^{t}. \end{cases} \tag{1}$$

In the following we aim to find particular solutions to equations in (1) one by one. Firstly, since $e^{-2t} \notin S$, we take $Y_1 = Ae^{-2t}$ for some constant $A \in \mathbb{R}$, then

$$Y_1' = -2Y_1, \quad Y_1'' = 4Y_1$$

and

$$4Y_1 - 4Y_1 - 3Y_1 = -3Ae^{-2t} = e^{-2t}.$$

So

$$A = -\frac{1}{3} \quad \text{and then} \quad Y_1 = -\frac{1}{3}e^{-2t}.$$

Secondly, since $e^t \in S$, we take $Y_2 = Bte^t$ for some constant $B \in \mathbb{R}$, then

$$Y_2' = Be^t + Bte^t, \quad Y_2'' = 2Be^t + Bte^t$$

and

$$2Be^t + Bte^t + 2(Be^t + Bte^t) - 3Bte^t = 4Be^t = e^t.$$

So

$$B = 1/4 \quad \text{and then} \quad Y_2 = \frac{1}{4}te^t.$$

Therefore we have got

$$Y(t) := Y_1(t) + Y_2(t) = -\frac{1}{3}e^{-2t} + \frac{1}{4}te^t.$$

is particular solution.

Step 3. (*General solution*) In conclusion, then general solution is

$$y(t) = c_1e^t + c_2e^{-3t} - \frac{1}{3}e^{-2t} + \frac{1}{4}te^t.$$

(b) By **Q**1(b), $S := \{e^{-2t}, te^{-2t}\}$ is the fundamental solutions set to the homogeneous equation. Now consider a particular solution $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1'' + 4Y_1' + 4Y_1 = e^{-2t}; \\ Y_2'' + 4Y_2' + 4Y_2 = t^2. \end{cases} \tag{2}$$

Firstly, since $e^{-2t} \in S$ and -2 is the solution to the characteristic equation with multiplicity 2, we take $Y_1(t) = At^2e^{-2t}$. Moreover,

$$Y_1'(t) = 2Ate^{-2t} - 2At^2e^{-2t}, \quad Y_1''(t) = 2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t}.$$

Then the first equation in (2) can be reduced to

$$2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t} + 4(2Ate^{-2t} - 2At^2e^{-2t}) + 4At^2e^{-2t} = 2Ae^{-2t} = e^{-2t}.$$

So

$$A = 1/2 \quad \text{and then } Y_1(t) = \frac{1}{2}t^2e^{-2t}.$$

Secondly, we take $Y_2(t) = B_1t^2 + B_2t + B_3$, then

$$\begin{aligned} t^2 &= 2B_1 + 4(2B_1t + B_2) + 4(B_1t^2 + B_2t + B_3) \\ &= 4B_1t^2 + (4B_2 + 8B_1)t + 2B_1 + 4B_2 + 4B_3. \end{aligned} \tag{3}$$

Then

$$B_1 = 1/4, B_2 = -1/2, B_3 = 3/8 \quad \text{and then} \quad Y_2 = \frac{1}{4}t^2 - \frac{1}{2}t + \frac{3}{8}.$$

In conclusion, the general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{2}t^2 e^{-2t} + \frac{1}{4}t^2 - \frac{1}{2}t + \frac{3}{8}$$

for some constants c_1, c_2.

(c) It is easy to see $\lambda = \pm 2i$ are solutions to the characteristic equation

$$\lambda^2 + 4 = 0.$$

Hence $S := \{\cos 2t, \sin 2t\}$ is the fundamental set of solutions. In the following, we construct the particular solution $Y(t) = Y_1(t) + Y_2(t)$.

$$\begin{cases} Y_1'' + 4Y_1 = \cos 3t, \\ Y_2'' + 4Y_2 = \sin 2t. \end{cases} \tag{4}$$

Since $\cos 3t \notin S$, take $Y_1(t) = A_1 \cos 3t + A_2 \sin 3t$, then

$$Y_1' = -3A_1 \sin 3t + 3A_2 \cos 3t, \quad Y_1'' = -9A_1 \cos 3t - 9A_2 \sin 3t.$$

So

$$Y_1'' + 4Y_1 = -5A_1 \cos 3t - 5A_2 \sin 3t = \cos 3t.$$

Hence

$$A_1 = -1/5, A_2 = 0 \quad \text{and then} \quad Y_1 = -\frac{1}{5}\cos 3t.$$

Secondly, since $\sin 2t \in S$, take $Y_2 = B_1 t \cos 2t + B_2 t \sin 2t$, then

$$\begin{aligned} Y_2' &= B_1 \cos 2t - 2B_1 t \sin 2t + B_2 \sin 2t + 2B_2 t \cos 2t, \\ Y_2'' &= -4B_1 \sin 2t - 4B_1 t \cos 2t + 4B_2 \cos 2t - 4B_2 t \sin 2t \end{aligned}$$

So

$$Y_2'' + 4Y_2 = -4B_1 \sin 2t + 4B_2 \cos 2t = \sin 2t.$$

Hence

$$B_1 = -1/4, B_2 = 0 \quad \text{and then} \quad Y_2 = -\frac{1}{4}t\cos 2t.$$

Combining all the above arguments implies the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{5}\cos 3t - \frac{1}{4}t\cos 2t.$$

(d) $S := \{e^t, e^{-3t}\}$ is the fundamental set of solutions. In the following, we construct a particular solution $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$ with

$$\begin{cases} Y_1'' + 2Y_1' - 3Y_1 = e^t \sin 2t, & \notin S \\ Y_2'' + 2Y_2' - 3Y_2 = te^t, & e^t \in S \\ Y_3'' + 2Y_3' - 3Y_3 = t\sin t. & \notin S \end{cases} \tag{5}$$

Take

$$Y_1 = e^t(A_1 \sin 2t + A_2 \cos 2t); \quad Y_2 = (B_1 t + B_2)te^t; \quad Y_3 = t(C_1 \sin t + C_2 \cos t).$$

Substitute into (5) and solve these constants above. I omit it here. The general solution is in the form

$$y(t) = c_1 e^t + c_2 e^{-3t} + Y_1 + Y_2 + Y_3.$$

□

Q3. Let $g(t)$ be an general function of t. Solve the non-homogeneous equation

$$y'' + 4y' + 4y = g(t).$$

Solution. By **Q1**(b), we know

$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$

forms a fundamental set of solutions to the corresponding homogeneous equation. We know the general form of the solution can be written as

$$y = u_1(t)y_1(t) + u_2(t)y_2(t),$$

with (u_1, u_2) solving

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}. \tag{6}$$

The above system gives

$$u_1'(t) = -tg(t)e^{2t}, \quad u_2'(t) = g(t)e^{2t}. \tag{7}$$

Integrating implies

$$u_1(t) = -\int tg(t)e^{2t}\,\mathrm{d}t, \quad u_2(t) = \int g(t)e^{2t}\,\mathrm{d}t.$$

□

Remark 1. *In general we choose* (u_1, u_2) *satisfying* (6), *but this is not necessary. You can also choose* $u_1 = u_1(t)$ *and* $u_2 \equiv 0$ *in this case, which is the so-called reduction method. Try to do it by yourself.*

Q4. (*Challenging problem*) Verify that $y_1 = t$ is a solution to

$$t^2 y'' - t(t+2)y' + (t+2)y = 0.$$

Find the general solution to

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0.$$

Hint: Firstly, use reduction method or Wronskian to find the other solution $y_2(t)$ to the homogeneous equation. Then use the "variation of parameters" or "reduction of order" to find the general solution.